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Teaching the Shakespeare of Mathematics

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During my year as President of The Mathematical Association the government began a review of the curriculum across all subjects taught in school. Given the constant tinkering with the education system by every government, this is probably a sentence that any MA president could write during their tenure. But 2013 saw government really trying to lay out their vision for what education should deliver in the coming decade.

In English, an aim of the curriculum review is to develop a student's love of literature through widespread reading for enjoyment. Students will get to read Shakespeare, romantic poetry, the great novels of the nineteenth century together with seminal works of world literature. It is hoped that this will expose students to the great works of literature and allow them to acquire an appreciation of 'our rich and varied literary heritage' [1, p. 13].

In science, by building on fundamental ideas and concepts, the curriculum review aims to foster in students 'a sense of excitement and curiosity about natural phenomena' [1, p. 99]. Students will study the fundamental mechanics of the cell, learn about stem cells, photosynthesis and genomics; they will be exposed to radiation (in a theoretical sense) and the evidence for the Big Bang. Such fundamental ideas will give students the scientific knowledge to be able to understand the uses and implications for the impact of science on society.

The document outlining a vision for mathematics starts promisingly with a mission statement:

> ' *Mathematics is a creative and highly inter-connected discipline that has been developed over centuries, providing the solution to some of history's most intriguing problems. It is essential to everyday life, critical to science, technology and engineering, and necessary in most forms of employment. A high-quality mathematics education therefore provides a foundation for understanding the world, the ability to reason mathematically, and a sense of enjoyment and curiosity about the subject.*'
>
> [1, p. 53]

But the curriculum that is suggested feels like it only gets students half way on this exciting journey. The curriculum focuses on tools and utility, the grammar of maths with none of the big stories. Where are the Shakespeare and the Darwin in the mathematics curriculum? Of course one needs a facility with mathematical language to be able to appreciate these ideas but, without the ideas, there is little incentive for the student to dedicate their time to the technical side of their subject.

In English, we have two GCSE qualifications: English Language and English Literature. Maybe there is a place for a mathematical 'literature' qualification alongside the utilitarian curriculum that is served up in the current GCSE. It shouldn't be a GCSE for the clever kids but should be targeted at a broad range of students. What is its aim? To develop a student's love of mathematics through doing mathematics for enjoyment, to foster an appreciation of our rich and varied mathematical heritage (to steal a phrase from the English curriculum reveiw).

It was these big ideas after all rather than the utilitarian aspect of mathematics that drew most mathematicians and teachers of mathematics to the subject: ideas of infinity, patterns in numbers, the attraction of symmetry, the power of proof. These are some of the themes which have the potential to instil excitement and curiosity in students studying mathematics.

The mission statement for the mathematical curriculum reform has in it some of the central themes that should inform such a mathematical literacy education: the historical context of the development of mathematical ideas, the impact of mathematics on the development of innovative technology, the interconnected nature of mathematics to culture and the humanities, the creativity and enjoyment that fuels mathematical development.

Just as in English a student isn't meant to grasp the full complexity of a Shakespeare play, we should be prepared to take the risky step of teaching big ideas that a student might not fully comprehend but rather they should be given a way to glimpse something of these great stories. Just as any course in English literature can give just a taste of the great works, a mathematical literacy course would not aim to be complete but to expose students to a sample of what is out there.

So what would the *Othello* or the *Animal Farm* of mathematics be? I have taken one particular story created by the person I regard as the Shakespeare of Mathematics: Bernhard Riemann. Other great stories can be found in my book *The Number Mysteries* [2] which in some sense I regard as my manifesto for what might be taught in a mathematical literature course.

Getting a student to see in four dimensions

I can still remember the excitement I felt the day I first ‘saw’ in four dimensions. Although I didn't physically ‘see’ four-dimensional shapes I had learnt the language that allowed me to conjure up these shapes in the mind's eye. But it is a language that we already teach students in school: the language of coordinates. So is it such a difficult challenge to teach kids to see in four dimensions?

These ‘four-dimensional goggles’ are possible thanks to a dictionary, invented by Descartes, which changes shapes into numbers. He developed the dictionary because of his frustration with how difficult it is to pin down the visual world. As he used to say: ‘Sense perceptions are sense deceptions’.

When we look up the location of a town in an atlas we find it identified by a two number grid location. A GPS can pinpoint very accurately every location on the earth using a pair of numbers. These numbers pinpoint our North-South, East-West location from a point on the equator that lies directly south of Greenwich in London.

For example, the University of Loughborough, the venue of the 2013 Mathematical Association Annual Conference, is on a latitude of 52.77 degrees north and 1.22 degrees west. In Descartes' dictionary we can translate the geographical location of the Conference into two numbers or coordinates $(-1.22, 52.77)$.

We can use a similar process to describe mathematical shapes. For example if we want to describe a square in this system of coordinates we can say that it is a shape with four vertices located at the points (0,0), (1,0), (0,1) and (1,1). Each edge corresponds to choosing two vertices where the coordinates are different in one position. For example one of the edges corresponds to the two coordinates (0,1) and (1,1). Here the first coordinate is different.

The flat, two-dimensional world needs just two coordinates to locate each position, but if we also want to include our height above sea-level then we could add a third coordinate. We will need this third coordinate too if we want to describe a three-dimensional cube in terms of coordinates. Its eight vertices can be described by the coordinates (0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1) and finally the extremal point at (1,1,1).

Again an edge consists of two points whose coordinates are different at exactly one position. If you look at a cube then you can easily count how many edges there are. But if you didn't have this picture, you could just count how many pairs of points there are which differ at one coordinate. Keep this in mind as we move to a shape where we don't have a picture.

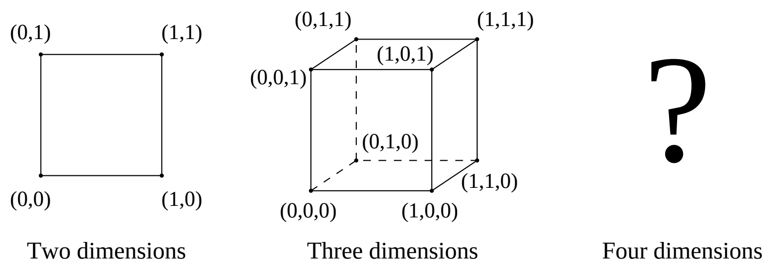

FIGURE 1

Descartes' dictionary has shapes and geometry on one side and, on the other, numbers and coordinates. The trouble is that the visual side runs out if we try to go beyond three-dimensional shapes since there isn't a fourth physical dimension where we can see higher dimensional shapes. (The beauty of Descartes' dictionary is that the other side of the dictionary keeps going beyond three dimensions.) To describe a four-dimensional object we just add a fourth coordinate which would just keep track of how far you are moving in this new direction. So although we can never physically build a four-dimensional cube, by using numbers we can still describe it precisely.

It has 16 vertices starting at (0,0,0,0) then extending to points at (1,0,0,0), (0,1,0,0) and stretching out to the furthest point at (1,1,1,1). The numbers are a code to describe the shape. And using this code we can explore the shape without ever having to physically see it.

For example, how many edges does this four-dimensional cube have? An edge corresponds to two points where one of the coordinates is different. Out of each vertex there are four edges, corresponding to changing each of the four coordinates one at a time. That looks as if there might be 16×4 edges. But notice that we actually count each edge twice this way: once as an edge emerging from the point at one end and also as an edge emerging from the point at the other end of the edge. So the total number of edges in the four-dimensional cube is $(16 \times 4)/2 = 32$.

And it doesn't stop there. You can move into five, six or even higher dimensions and build hypercubes in these worlds. For example a hypercube in N dimensions will have 2^N vertices. Out of each of these vertices there will be N edges emerging, each of which we are counting twice. So the N-dimensional cube has $N \times 2^{N-1}$ edges.

Mathematics gives you a sixth sense which means that you can play with these shapes that live beyond the limits of our three-dimensional universe.

The History

One of the key components I think that is missing from mathematical education both at school and at university is placing mathematical discoveries in a historical context. It is really important that students understand that mathematics is created/discovered by people. It isn't handed down in some huge mathematical textbook from on high. To understand and appreciate how mathematicians before them struggled with new ideas is empowering for a student who is wrestling with new ideas in the classroom.

It was the nineteenth-century German mathematician Bernhard Riemann, the Shakespeare of Mathematics, who realized the power of Descartes's dictionary to take you into these strange new worlds. It is interesting how these new ideas grew out of a corresponding shift in the educational philosophy in Germany. France, which had been the powerhouse of mathematical innovation at the end of the eighteenth century, regarded mathematics as a tool to serve the government. But Humboldt's educational reforms in Germany at the beginning of the nineteenth century shifted this focus.

Humboldt placed the emphasis on serving the needs of the individual, rather than the state. Utilitarian goals were to be replaced by a desire to gain knowledge for its own sake. All this was in stark contrast to Napoleon's utilitarian view of the subject. In 1830, Carl Jacobi, a leading professor in Berlin, wrote to a colleague in Paris boasting about the superiority of the German approach to mathematics.

‘ … the sole object of science is honour of the human spirit ……

> a problem in the theory of numbers is worth as much as a problem of the system of the world.'

I often get the feeling that we are still stuck in Napoleonic France, just doing mathematics to serve the state. If only we could find a modern-day Humboldt to take the reins at the Department of Education ….

For the first time in Germany, the study of mathematics formed a major part of the curriculum in the new schools and universities. And mathematicians, freed from the need to model the physical world, began instead to explore mathematical ideas for their own sake. It gave rise to the creation of geometry that lives beyond our three-dimensional universe.

But this does not mean that the mathematics that was discovered by Riemann had no impact on the world. Sometimes that impact is unexpected and could never have been anticipated in advance. The geometry of hyperspace was the key to Einstein's breakthroughs in the theory of relativity. It gave him the arena to unify time and space. And without Einstein's breakthroughs on how time and space are intimately connected we could never have created the GPS that helped some of us to find our way to the conference in Loughborough. But the mathematics of hyperspace hasn't just had an impact on technology.

The Art

So often students move from one classroom to another at school not realising the connections and synergies that exist between all the subjects taught in school. They go from the mathematics class, to the history class, to the art class, to the music class, oblivious to the fact that there is any connection between the subjects they are learning. Understanding how new mathematical ideas have often been a great catalyst for new art is a possible way to engage a whole new audience in the school who feel disenfranchised by the sometimes very technical nature of mathematics.

The concept of a four-dimensional cube was central to one of the iconic modern buildings on the Paris skyline. To celebrate the 200th anniversary of the French Revolution, President François Mitterrand commissioned the Danish architect Johan-Otto von Spreckelsen to build something special at La Défense, the financial district of Paris. The building would line up with several other significant Parisian buildings: the Louvre, the Arc de Triomphe and the Egyptian Needle in what has become know as the *Mitterrand Perspective*.

The architect certainly didn't disappoint. He built a huge arch which weighs a staggering 300,000 tonnes and is so large that the towers of Notre Dame would fit through the middle. Unfortunately von Spreckelsen died two years before the completion of the arch. Perhaps less well known to the Parisians who see it every day is the fact that he actually built a four-dimensional cube in the heart of their capital.

Well, it isn't quite a four-dimensional cube since we only live in a three-dimensional universe − it is in fact the shadow of a four-dimensional cube.

Just as the Renaissance artists were faced with the challenge of painting three-dimensional shapes on a flat two-dimensional canvas, the architect at La Défense has tried to capture a shadow of the four-dimensional cube in our three-dimensional world. To create the illusion of seeing a three-dimensional cube while looking at a two-dimensional canvas, the artist might draw a square inside a larger square and then join up the corners of the squares to complete the picture of the cube. Of course it's not really a cube but it retains enough information to see all the edges for example.

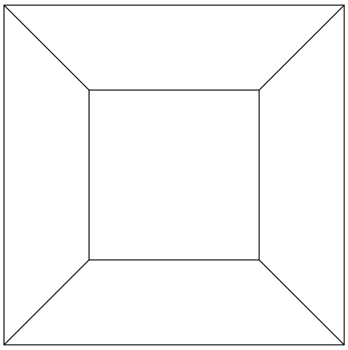

FIGURE 2

The architect at La Défense used the same idea to build a projection of a four-dimensional cube in three-dimensional Paris. The arch consists of a small cube sitting inside a larger cube with edges joining up the vertices of the smaller and larger cubes. If you count carefully you can see the 32 edges that we identified using Descartes' coordinates.

FIGURE 3: Picture of La Grande Arche

Whenever I've visited La Grande Arche at La Défense, it is uncanny how there is always a howling wind which seems to suck you through the centre of the cube. So serious has this wind become that the designers were forced to erect a canopy at the heart of the arch to disrupt the flow of air. It's almost as if constructing a shadow of a hypercube in Paris has opened up a portal to another dimension.

There are other ways to get a feel for the four-dimensional cube in our three-dimensional world. For example you could unwrap it. Think of the way that you would build a three-dimensional cube out of a piece of two-dimensional card. First you would draw six squares connected in a cross shape, one square for each face of the cube, then by wrapping the cross shape up you can build a cube. In a similar fashion, it is possible in our three-dimensional world to build a 3D net which, if you had a fourth dimension, could be wrapped up to make a 4D cube.

To make your own 4D cube cut out and assemble eight cubes. These will be the 'faces' of your 4D cube. To make the net of the 4D cube you need to join these eight cubes together. Start by gluing together the first four cubes into a column, one stacked upon the other. Next take the remaining four cubes and stick them to the faces of one of the four cubes in the column. Your unwrapped hypercube looks like two intersecting crosses. To fold this thing up you would need to start by joining up the bottom and top column of cubes. The next step would be to join the outward facing squares of two of the cubes stuck on either side of the column to the bottom cube in the tower. Then finally you'd need to glue the faces of the other two cubes to the remaining two faces of the bottom cube. If you tried to glue this thing together, you'd soon get in a tangle as there just isn't enough room in three dimensions: you need a fourth dimension to do this.

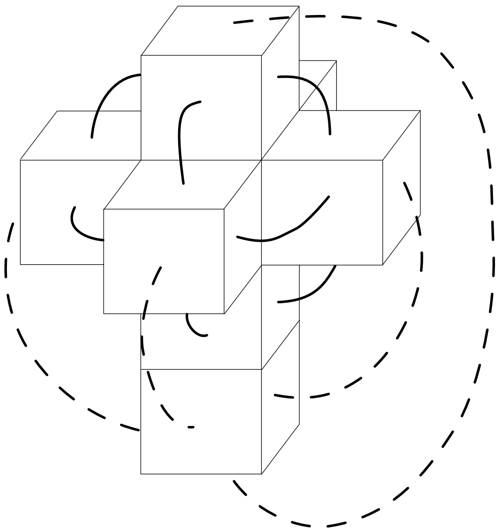

FIGURE 4

Just as the architect in Paris was inspired by the shadow of the 4D cube, the artist Salvador Dali was intrigued by the idea of this unwrapped hypercube. In his painting *Crucifixion* (*Corpus Hypercubus*) Dali depicts Christ being crucified on the 3D net of a four-dimensional cube. The idea of the fourth dimension being something beyond our material world resonated for Dali with the spiritual world beyond our physical universe. The unwrapped hypercube consists of two intersecting crosses and the picture suggests that his ascension to heaven is connected with trying to wrap this 3D structure up into a fourth dimension that transcends physical reality.

Whatever way we try to depict these 4D shapes in our three-dimensional universe they can never show a complete picture just as a shadow or silhouette in the two-dimensional world can only give partial information. As we move and turn the object, the shadow changes, but you never see everything. This theme was picked up by novelist Alex Garland in his book *The Tesseract*, another name for a four-dimensional cube. The narrative describes the views of different characters in the central story set within the gangster world of Manila. No one narrative provides a clear picture but by piecing together all the strands, like looking at the many different shadows cast by a shape, you start to understand what the story might possibly be.

Revealing how artists have used mathematical structures in their work provides an intriguing way to access the subject for those who might at first think of themselves more on the humanities side of the two cultures. But that's the point. Thinking in terms of two cultures was always a false dichotomy. We need to find more ways to break down the walls between different classrooms.

The Application

One of the key themes picked up in the curriculum mission statement is the importance of mathematics to the development of new technology. I have already alluded to the role hyperspace played in the development of the GPS. But every time information is converted into digital data, those series of 0s and 1s can be interpreted as a corner of a very high dimensional cube. Mathematics has played a central role in creating clever ways of encoding data in high dimensions so that any errors that might occur while the information is in transit can be corrected by the person receiving the message.

The mathematics of error-correcting codes is not complicated and again could form an interesting application of high-dimensional geometry that is accessible to a GCSE student. For example suppose a message you want to send consists of the digital string

$$(0,0,1,1,1,1,1,0,1,1,0,0,0,1,0,1,0,0,1,1,1,1,1,1,1).$$

One can break this up into five short messages which represent corners on a five-dimensional cube:

0	0	1	1	1
1	1	0	1	1
0	0	0	1	0
1	0	0	1	1
1	1	1	1	1

Suppose that, during transmission of the message, one of the digits gets flipped. There is no way to tell which digit was corrupted, but a simple trick can help preserve the data. This consists of adding an extra row and column to the grid which record information about how many zeros and ones there are in it. Count how many 1s there are in each row and each column, starting with the first column. If there are an odd number of 1s in this first column then the extra digit you add to the bottom of this column is a 1. If there are an even number of 1s (0 is an even number in this setting) then put a 0 at the end of this column. Proceed along the five columns adding digits at the bottom of each column to keep track of the number of 1s in each column. Do the same for the rows.

0	0	1	1	1	1
1	1	0	1	1	0
0	0	0	1	0	1
1	0	0	1	1	1
1	1	1	1	1	1
1	0	0	1	0	

In the bottom right-hand square place a 0 or a 1 corresponding to whether the column above it has an even or odd number of 1s. Interestingly this will also record the parity of the number of 0s and 1s in the bottom row too. The trick is to realise that actually this number tells you whether there is an odd or even number of 1s in the whole 5×5 grid.

0	0	1	1	1	1
1	1	0	1	1	0
0	0	0	1	0	1
1	0	0	1	1	1
1	1	1	1	1	1
1	0	0	1	0	0

Now the message is ready to send. Suppose during transmission one of the digits gets changed. It is now possible to identify which digit got corrupted. Look at the original 5×5 block of numbers and see whether there is an odd or even number of 1s and compare it with the 0s and 1s you added, remembering that these kept track of the number of odd or even 1s in each column. If the error occurred in one of the numbers in the 5×5 grid then there will be one row and one column where the numbers you added don't match up with the number of 1s in that column and row. Look at where the column and row intersect and there is the location of the error that occurred in the transmission.

For example, can you identify which number has been changed in the following set-up? I have put in dark lines to separate the 5×5 grid containing the message and the extra numbers we've added.

0	0	1	1	1	1
0	1	0	1	1	0
0	0	0	1	0	1
1	0	0	1	1	1
1	1	1	1	1	1
1	0	0	1	0	0

The first column in the 5×5 grid has an even number of 1s. But the digit we added at the bottom of this column was a 1 indicating there were originally an odd number, so the first column contains the error.

Now turn to the rows. It is in the second row where things don't match up. There are an odd number of 1s in the second row of the 5×5 grid but your check digit indicates that there should be an even number. So now we can identify that the error occurred in the first column, second row.

If the error actually occurred in one of the check digits you added then the entry in the lower right hand column will not match the last row or the last column. If it doesn't match the last row for example, you'll know that one of the entries in the last row has been changed so just check each of the columns to see which column doesn't match up. If you find actually it's the sixth column that doesn't match then actually it's the number in the lower right hand corner that got corrupted.

Error-correcting codes like this are used in everything from CDs to satellite communications. Every time computers talk to each other they are using these ideas to make sure interference doesn't corrupt the messages that are sent.

More Big Stories

For me this is just one example of the sort of topic that could be covered in a Mathematical Literature course. None of the mathematics is out of reach yet the ideas are big. Here are some other ideas that could be seeds for other big stories that might form part of such a course:

(1) What is the next number in this sequence: 1, 1, 2, 3, 5, 8, 13 …? The Fibonacci numbers never fail to intrigue students. They are key, not only to the mating habits of rabbits, but musical rhythms, growing shells and Le Corbusier's buildings.

(2) What are the chances that two children in a class have the same birthday? If the class has at least 23 children it is more than 50 : 50. The mathematics of probability is as counter-intuitive as it is useful in navigating the risks that surround us.

(3) The mathematics of fractal shapes explains why the coastline of Britain could be infinite, why a Jackson Pollock is so special and why Nature is full of such complexity.

(4) The proof that there are infinitely many primes is as simple as it is devastating in its implications yet a million dollars awaits the person who can unlock a way to navigate a way through these numbers.

(5) There is more than one infinity and some infinities are bigger than others.

(6) The population growth of lemmings, the weather, and the stability of the solar system are all at the mercy of a simple mathematical equation which give rise to chaos, one of the most important mathematical stories of the last century.

(7) From the Sage at Gateshead to the Guggenheim in Bilbao, mathematical shapes are key to creating the innovative architecture of the modern skyline.

(8) Why are the geometries of the bagel and the coffee cup related? It is because you can mould one into the other. The mathematics of topology or bendy geometry is a new way at looking at shapes.

(9) The mathematics of symmetry explains why there are only five different dice and seventeen different symmetries on the walls in the Alhambra.

I am not an educationalist. I am a mathematician. I don't pretend to be able to advise on how mathematics should be taught and I know that training teachers to be able to tell these stories will be a big challenge. But I know what turned me on to the subject. It was being shown what mathematics is really about. Real mathematics. It was being exposed to the big stories, the Shakespeare of Mathematics that inspired me. Even if a student chooses not to become a mathematician I still believe that we should be braver in the mathematics to which we expose our students. It is these big ideas not the technical side of the subject that I believe will realise the dream of the curriculum mission statement to instil in students ‘*a sense of enjoyment and curiosity about the subject*’.

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